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Position-dependent effective mass Dirac equations with *PT*-symmetric and non-*PT*-symmetric potentials

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Abstract
We present a new procedure to construct the one-dimensional non-Hermitian imaginary potential with a real energy spectrum in the context of the position-dependent effective mass Dirac equation with the vector-coupling scheme in 1+1 dimensions. In the first example, we consider a case for which the mass distribution combines linear and inversely linear forms, the Dirac problem with a *PT*-symmetric potential is mapped into the exactly solvable Schrödinger-like equation problem with the isotonic oscillator by using the local scaling of the wavefunction. In the second example, we take a mass distribution with smooth step shape, the Dirac problem with a non-*PT*-symmetric imaginary potential is mapped into the exactly solvable Schrödinger-like equation problem with the Rosen–Morse potential. The real relativistic energy levels and corresponding wavefunctions for the bound states are obtained in terms of the supersymmetric quantum mechanics approach and the function analysis method.

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1. Introduction

After Bender and Boettcher [1] for the first time investigated a non-Hermitian complex potential with a real energy spectrum on the *PT*-symmetric quantum mechanics, there has been growing interest in studying the non-Hermitian complex potentials in the setting of the non-relativistic Schrödinger equation with a constant mass [2–4] and position-dependent effective mass Schrödinger equation [5]. A potential $V(x)$ is said to possess *PT* symmetry if the relation $V(-x) = V*(x)$ or $V(\xi - x) = V*(x)$ exists under the transformation of $x \to -x$ (or $x \to \xi - x$) and $i \to -i$, where P denotes parity operator (space reflection) and T denotes time reversal. Non-Hermitian potential models have many applications in different research areas, for example, in the study of nuclear physics [6], quantum field theories [7] and electromagnetic wave travelling in a planar slab waveguide [8].

Recently, some authors [9–13] have investigated the Klein–Gordon equation and Dirac equation with a constant mass in the context of *PT* symmetry [1] and pseudo-Hermiticity [3]. Mustafa [9] studied the exact energies for Klein–Gordon particle and Dirac particle with a constant mass in the generalized complex Coulomb potential. Znojil [10] analysed the Klein–Gordon equations in the setting of pseudo-Hermiticity. With the help of the Nikiforov–Uvarov method which is based on reducing a second-order linear differential equation to a generalized equation of hypergeometric type, Egrifes *et al* [11] investigated the bound states of the Klein–Gordon and Dirac equations with a constant mass for the one-dimensional generalized Hulthén potential within the framework of *PT*-symmetric quantum mechanics. In [12], the authors investigated the bound-state energy equation for the *PT*-symmetric versions of the Rosen–Morse well and Scarf II potential in the Klein–Gordon theory with equally mixed potentials. Sinha and Roy [13] investigated the one-dimensional solvable Dirac equation with non-Hermitian scalar and pseudoscalar interactions, possessing real energy spectra. For the relativistic wave equations, one of recent developments is to study the non-Hermitian complex potentials in the context of Klein–Gordon equation and Dirac equation with a constant mass, another is to investigate the position-dependent effective mass Klein–Gordon equations and Dirac equations for Hermitian potentials. It is usually expected that, in the relativistic ambiance, the ordering ambiguity of the mass and momentum operators should disappear. Notwithstanding, it is important to remember that there are difficulties to define consistently from first principles the Dirac equation for fermions, and the Klein–Gordon one for bosons, when we take into account spacetime dependence for the mass of the particle. This happens due to the fact that physical particles in quantum field theory must belong to an irreducible representation of the Poincarè algebra [14, 15]. One should be able to find generators specifying the particle properties, usually its mass and helicity. However, it is quite difficult to accomplish with this task in the case of spatially dependent masses. So, we think that one should keep in mind that all of these usually thought as relativistic equations for position-dependent masses should be taken as effective equations. However, this ordering ambiguity is certainly present in the non-relativistic case [16]. In this regard, Alhaidari [17] studied the exact solution of the three-dimensional Dirac equation for a charged particle with spherically symmetric singular mass distribution in the Coulomb field. Vakarchuk [18] investigated the exact solution of the Dirac equation for a particle whose potential energy and mass are inversely proportional to the distance from the force centre. In [19], the authors considered the smooth step mass distribution and solved approximately the one-dimensional Dirac equation with the spatially dependent mass for the generalized Hulthén potential. In [20], the authors investigated the exact solution of the one-dimensional Klein–Gordon equation with the spatially dependent mass for the inversely linear potential. As far as we know, there are only few contributions that give the solutions of the position-dependent effective mass relativistic wave equations for some Hermitian potentials. Therefore, it is of considerable interest to investigate the solution of the effective mass Klein–Gordon equation or effective mass Dirac equation for a non-Hermitian complex potential with a real energy spectrum.

In this work, we propose a method to construct a one-dimensional exactly solvable non-Hermitian potential with a real energy spectrum in the setting of the one-dimensional Dirac equation with the vector-coupling scheme in the presence of position-dependent mass. In order to illustrate the scheme, we give two examples. In the first example, we consider the mass with linear and inversely linear forms in one spatial dimension and choose the smooth step mass distribution in another example. With the help of the supersymmetric quantum mechanics approach and the function analysis method, we give the relativistic energy levels and corresponding spinor wavefunctions for the bound states.

2. One-dimensional Dirac equation with position-dependent mass

Choosing the atomic units $h/2\pi = \hbar = c = 1$, the one-dimensional time-independent Dirac equation with any given interaction potential $V(x)$ in the vector-coupling scheme is given by [19, 21]

$$\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + (E - V(x))\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - M\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\Psi(x) = 0, \tag{1}$$

where E denotes the energy and M denotes the mass. The spinor wavefunction $\Psi(x)$ has two components. We denote the upper and lower components by $\phi(x)$ and $\theta(x)$, respectively. Equation (1) can be decomposed into the following two coupled differential equations:

$$-\mathrm{i}\frac{\mathrm{d}\theta}{\mathrm{d}x} + [E - V(x)]\theta - M(x)\phi = 0, \tag{2}$$

$$\mathrm{i}\frac{\mathrm{d}\phi}{\mathrm{d}x} + [E - V(x)]\phi - M(x)\theta = 0. \tag{3}$$

Eliminating the lower spinor component form equations (2) and (3), we obtain a second-order differential equation, which contains first-order derivatives:

$$\begin{aligned}-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} &+ \left[2EV(x) - V^2(x) - \mathrm{i}\frac{\mathrm{d}V(x)}{\mathrm{d}x} - \mathrm{i}\frac{1}{M(x)}\frac{\mathrm{d}M(x)}{\mathrm{d}x}(E - V(x))\right]\phi\\ &+ \frac{1}{M(x)}\frac{\mathrm{d}M(x)}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x} = [E^2 - M^2(x)]\phi.\end{aligned} \tag{4}$$

Now, we perform the following local scaling of the wavefunction:

$$\phi(x) = \sqrt{M(x)}\varphi(x). \tag{5}$$

Substituting expression (5) into equation (4), we obtain the following Schrödinger-like equation satisfied by the new wavefunction $\varphi(x)$:

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + V_{\mathrm{eff}}(x)\varphi = E^2\varphi, \tag{6}$$

where $V_{\mathrm{eff}}(x)$ is defined as

$$V_{\mathrm{eff}}(x) = -V^2 - \mathrm{i}\frac{\mathrm{d}V}{\mathrm{d}x} + M^2 + \mathrm{i}\left(\frac{\frac{\mathrm{d}M}{\mathrm{d}x}}{M}\right)V + E\left(2V - \mathrm{i}\left(\frac{\frac{\mathrm{d}M}{\mathrm{d}x}}{M}\right)\right) - \frac{1}{2}\left(\frac{\frac{\mathrm{d}^2M}{\mathrm{d}x^2}}{M}\right) + \frac{3}{4}\left(\frac{\frac{\mathrm{d}M}{\mathrm{d}x}}{M}\right)^2. \tag{7}$$

Here, we impose that the vector potential has the form

$$V(x) = \frac{\mathrm{i}}{2}\frac{1}{M(x)}\frac{\mathrm{d}M(x)}{\mathrm{d}x}. \tag{8}$$

This is a non-Hermitian complex potential. Substituting equation (8) into equation (7) and making some straightforward calculations, we obtain the effective potential in equation (6):

$$V_{\mathrm{eff}}(x) = M(x)^2. \tag{9}$$

At this point we note that one can obtain a usual bound-state energy spectrum from a system with an imaginary potential and even with a mass with negative values along the spatial axis.

Given a position-dependent mass distribution function $M(x)$, we can produce a non-Hermitian imaginary potential by using equation (8) and solve the Schrödinger-like equation (6) to obtain the relativistic energy spectra in the setting of the Dirac equation with an imaginary potential and position-dependent mass.

3. Examples

3.1. Linear and inversely linear mass distribution

For the first example, we choose the mass distribution with linear and inversely linear forms in one spatial dimension:

$$M(x) = M_0\left(\frac{|x|}{d} + \frac{d}{|x|}\right), \qquad M_0, d > 0, \tag{10}$$

where d is a constant with space dimension. The mass varies from the value $M = +\infty$ for $x = -\infty$ to the value $M = 2M_0$ for $x = -d$ and to the value $M = +\infty$ for $x = 0$, and also changes from the value $M = +\infty$ to the value $M = 2M_0$ and to the value $M = +\infty$ in the range of $0 \leqslant x < +\infty$. With the help of equation (8), we have an imaginary potential given by

$$V(x) = \frac{\mathrm{i}}{2}\frac{(x^2 - d^2)}{x(x^2 + d^2)}. \tag{11}$$

This potential is singular at $x = 0$. The imaginary potential (11) shows $V(-x) = V*(x)$, thus, this potential is PT-symmetric. Substituting equation (10) into equation (9) and using equation (6), we obtain a Schrödinger-like equation

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + M_0^2\left(\frac{x^2}{d^2} + \frac{d^2}{x^2}\right)\varphi = \tilde{E}\varphi, \tag{12}$$

where $\tilde{E} = E^2 - 2M_0^2$. Equation (12) shows that the effective mass one-dimensional Dirac equation with the PT-symmetric potential (11) can be mapped into the exactly solvable Schrödinger-like equation with the potential of a harmonic oscillator plus a centrifugal barrier in one-dimensional space. For the second-order differential equation with the form of equation (12), we [20] have dealt with it by using the supersymmetric quantum mechanics method [22] and the shape-invariance approach [23] in the setting of Schrödinger equation for the isotonic harmonic oscillator and in the setting of Klein–Gordon equation with position-dependent mass for the inversely linear scalar potential. We write the ground-state wavefunction $\varphi_0(x)$ in the form

$$\varphi_0(x) = \exp\left(-\int W(x)\,\mathrm{d}x\right), \tag{13}$$

where $W(x)$ is called a superpotential in supersymmetric quantum mechanics. Substituting equation (13) into equation (12) yields the following equation for $W(x)$:

$$W^2(x) - \frac{\mathrm{d}W(x)}{\mathrm{d}x} = M_0^2\left(\frac{x^2}{d^2} + \frac{d^2}{x^2}\right) - \tilde{E}_0, \tag{14}$$

where $\tilde{E}_0$ is the ground-state energy. Equation (14) is a nonlinear Riccati equation. Putting the superpotential $W(x)$ in the form

$$W(x) = Ax + \frac{B}{x}, \tag{15}$$

and substituting this expression into equations (13), we obtain the unnormalized ground-state wavefunction $\varphi_0(x)$:

$$\varphi_0(x) = x^{-B}\,\mathrm{e}^{-\frac{1}{2}Ax^2}. \tag{16}$$

In this work, we will deal with bound-state solutions, i.e., the wavefunction $\varphi_0(x)$ must satisfy the boundary conditions that $\varphi_0(x)$ is zero at $x = 0$ and $\varphi_0(x)$ becomes zero when $x \to \pm\infty$.

Thus, we have the restriction conditions: $B < 0$ and $A > 0$. Substituting equation (15) into equation (14), we obtain a set of equations

$$2AB - A = -\tilde{E}_0, \tag{17a}$$

$$A^2 = M_0^2 / d^2, \tag{17b}$$

$$B^2 + B = d^2 M_0^2. \tag{17c}$$

Solving equations (17a)–(17c), we obtain

$$\tilde{E}_0 = A - 2AB, \qquad A = M_0/d, \qquad B = \frac{-1 - \sqrt{1 + 4d^2 M_0^2}}{2}. \tag{18}$$

In terms of the superpotential $W(x)$ given in equation (15), we can construct the following two supersymmetric partner potentials:

$$V_{\text{eff}_+}(x) = W^2(x) + \frac{\mathrm{d}W(x)}{\mathrm{d}x} = 2AB + A + A^2x^2 + \frac{1}{x^2}(B^2 - B), \tag{19a}$$

$$V_{\text{eff}_-}(x) = W^2(x) - \frac{\mathrm{d}W(x)}{\mathrm{d}x} = 2AB - A + A^2x^2 + \frac{1}{x^2}(B^2 + B). \tag{19b}$$

These two partner potentials $V_{\text{eff}_+}(x)$ and $V_{\text{eff}_-}(x)$ possess the following relationship:

$$V_{\text{eff}_+}(x, a_0) = V_{\text{eff}_-}(x, a_1) + R(a_1), \tag{20}$$

where $a_0 = B$, a_1 is a function of a_0, i.e., $a_1 = f(a_0) = a_0 - 1$, and $R(a_1)$ is independent of x, $R(a_1) = 4A(a_0 - a_1)$. From equation (20), we know that the two partner potentials $V_{\text{eff}_+}(x)$ and $V_{\text{eff}_-}(x)$ have the similar shapes and they are shape-invariant potentials in the senses of [23]. For the potential $V_{\text{eff}_-}(x)$, we use the shape-invariance approach [23] to determine the energy spectra, which are given by

$$\tilde{E}_0^{(-)} = 0, \tag{21a}$$

$$\begin{aligned}\tilde{E}_n^{(-)} &= \sum_{k=1}^{n} R(a_k) = R(a_1) + R(a_2) + \cdots + R(a_n) \\ &= 4A(a_0 - a_1) + 4A(a_1 - a_2) + 4A(a_2 - a_3) + \cdots + 4A(a_{n-1} - a_n) \\ &= 4A(a_0 - a_n) \\ &= 4nA\end{aligned} \tag{21b}$$

where the quantum number $n = 0, 1, 2, \ldots$. From equations (14) and (19b), we have the following relation:

$$V_{\text{eff}}(x) = M_0^2\left(\frac{x^2}{d^2} + \frac{d^2}{x^2}\right) = V_{\text{eff}_-}(x, a_0) + \tilde{E}_0. \tag{22}$$

With the help of equations (18), (21) and (22), we obtain the solution for $\tilde{E}$ in equation (12):

$$\tilde{E} = \tilde{E}_0 + \tilde{E}_n^{(-)} = A - 2AB + 4nA. \tag{23}$$

Using the expressions $\tilde{E} = E^2 - 2M_0^2$ and applying equation (18) in equation (23), we find the following relativistic energy spectrum for the one-dimensional PT-symmetric potential (11) in the setting of the Dirac theory with the mass distribution form given in equation (10):

$$E_n = \pm\left[2M_0^2 + \frac{M_0}{d}\left(4n + 2 + \sqrt{1 + 4d^2M_0^2}\right)\right]^{1/2}. \tag{24}$$

In the case of the spatial dependence of the mass, we can regularize the problem of the Dirac equation with the PT-symmetric singular potential (11) and obtain the bound-state energy spectra, avoiding nonphysical divergences.

With the help of the superpotential $W(x)$ given in equation (15) and the ground-state wavefunction $\varphi_0(x)$ given in equation (16), one can obtain the unnormalized excited-state wavefunctions by using the operator approach proposed by Dabrowskafor *et al* [24]. The corresponding normalization coefficients can be obtained by applying the explicit recursion relations on the normalization coefficients of wavefunctions given in [25]. Fakhri and Chenaghlou [26] also constructed the recursion relations on the coefficients of associated hypergeometric functions (not the wavefunctions). Here, we use the conventional function analysis method to obtain the unnormalized excited wavefunctions. Substituting equation (23) into equation (12), it becomes the following equation:

$$-\frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2} + M_0^2\left(\frac{x^2}{d^2} + \frac{d^2}{x^2}\right)\varphi(x) = (4nA + A - 2AB)\,\varphi(x). \tag{25}$$

Writing the wavefunction $\varphi(x)$ as $\varphi(x) = x^{-B}\,\mathrm{e}^{-\frac{1}{2}Ax^2}L(x)$, equation (25) can be reduced to the following equation satisfied by $L(x)$:

$$\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2} - 2\left(Ax + \frac{B}{x}\right)L(x) + 4nAL(x) = 0. \tag{26}$$

Introducing the new variable $y = Ax^2$, equation (26) turns to the following form:

$$y\frac{\mathrm{d}^2L(y)}{\mathrm{d}y^2} + \left(-B - \frac{1}{2} + 1 - y\right)\frac{\mathrm{d}L(y)}{\mathrm{d}y} + nL(y) = 0, \tag{27}$$

Equation (27) is the well-known differential equation satisfied by the Laguerre polynomials $L_n^{(-B-1/2)}(Ax^2)$, hence the wavefunction $\varphi(x)$ can be expressed as

$$\varphi_n(x) = x^{-B}\,\mathrm{e}^{-\frac{1}{2}Ax^2}L_n^{(-B-1/2)}(Ax^2). \tag{28}$$

Applying equations (5) and (28) and making some parameter replacements given in equation (18), we obtain the unnormalized upper spinor wavefunction corresponding to energy level E_n:

$$\phi_n(x) = \sqrt{M_0\left(\frac{|x|}{d} + \frac{d}{|x|}\right)}\,x^{\frac{1+\sqrt{1+4d^2M_0^2}}{2}}\,\mathrm{e}^{-\frac{1}{2}\frac{M_0}{d}x^2}L_n^{(\frac{1}{2}\sqrt{1+4d^2M_0^2})}\left(\frac{M_0}{d}x^2\right). \tag{29}$$

By using the differential and recursion properties of the Laguerre polynomials, we can obtain the lower spinor wavefunction corresponding to energy level E_n from equation (3):

$$\begin{aligned}\theta_n(x) = \frac{1}{M_0\left(\frac{|x|}{d} + \frac{d}{|x|}\right)}\Bigg[&\left(E_n - \mathrm{i}\frac{M_0}{d}x + \mathrm{i}\left(2n + \frac{1}{2} + \frac{1}{2}\sqrt{1+4d^2M_0^2}\right)\frac{1}{x}\right)\phi_n(x)\\ &- 2\mathrm{i}\left(n + \frac{\sqrt{1+4d^2M_0^2}}{2}\right)\frac{1}{x}\phi_{n-1}(x)\Bigg].\end{aligned} \tag{30}$$

3.2. Smooth step mass distribution

In the next application we consider the smooth step mass [19],

$$M(x) = M_0(1 + \eta\tanh\alpha x), \tag{31}$$

where η is a small parameter and η satisfies $|\eta| < 1$. The mass increases from the value $M = M_0(1-\eta)$ for $x = -\infty$ to the value $M = M_0(1+\eta)$ for $x = +\infty$. The significant variations are occurring in the range of $-\frac{1}{\alpha} < x < \frac{1}{\alpha}$, i.e., $M(-1/\alpha) \cong M_0(1-0.762\eta)$, $M(1/\alpha) \cong M_0(1+0.762\eta)$. Substituting equation (31) into equation (8), we obtain a non-Hermitian imaginary potential

$$V(x) = \frac{\mathrm{i}}{2}\frac{\alpha\eta\,\mathrm{sech}^2\,\alpha x}{1+\eta\tanh\alpha x}. \tag{32}$$

With the help of equations (31), (9) and (6), we obtain a Schrödinger-like equation

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} - (V_1\,\mathrm{sech}^2\,\alpha x + V_2\tanh\alpha x)\varphi = \tilde{E}\varphi, \tag{33}$$

where we have defined the parameters $V_1 = \eta^2 M_0^2$, $V_2 = -2\eta M_0^2$ and $\tilde{E} = E^2 - M_0^2(1+\eta^2)$. Equation (33) shows that the effective mass one-dimensional Dirac equation with the non-Hermitian potential (32) can be mapped into the exactly solvable Schrödinger-like equation with the Rosen–Morse potential. We write the ground-state wavefunction $\varphi_0(x)$ in the form

$$\varphi_0(x) = \exp\left(-\int W(x)\,\mathrm{d}x\right). \tag{34}$$

Substituting equation (34) into equation (33), we obtain the following equation satisfied by the superpotential $W(x)$:

$$W^2(x) - \frac{\mathrm{d}W(x)}{\mathrm{d}x} = -(V_1\,\mathrm{sech}^2\,\alpha x + V_2\tanh\alpha x) - \tilde{E}_0, \tag{35}$$

where $\tilde{E}_0$ is the ground-state energy. Equation (35) is a nonlinear Riccati equation. Taking the superpotential $W(x)$ in the fashion,

$$W(x) = Q_1 + \frac{Q_2}{2}(1-\tanh\alpha x), \tag{36}$$

and substituting this expression into equation (35), we obtain

$$Q_1^2 = -\tilde{E}_0 - V_2, \tag{37a}$$

$$2Q_1Q_2 + 2\alpha Q_2 = -4V_1 + 2V_2, \tag{37b}$$

$$Q_2^2 - 2\alpha Q_2 = 4V_1. \tag{37c}$$

Solving the set of equations (37a)–(37c), we have

$$Q_1 = \frac{V_2}{Q_2} - \frac{Q_2}{2}, \tag{38a}$$

$$Q_2 = \alpha\left[1 - \sqrt{1+\frac{4V_1}{\alpha^2}}\right]. \tag{38b}$$

Substituting equation (36) into equations (34) and using equation (38a), we obtain the unnormalized ground-state wavefunction $\varphi_0(x)$:

$$\varphi_0(x) = \mathrm{e}^{-\frac{V_2}{Q_2}x}(\cosh\alpha x)^{\frac{Q_2}{2\alpha}}. \tag{39}$$

For the bound-state solutions, the wavefunction $\varphi_0(x)$ must satisfy the boundary condition that $\varphi_0(x)$ becomes zero when $x \to \pm\infty$. In order to make the wavefunction $\varphi_0(x)$ satisfy the regularity conditions, we can obtain from equation (39) that $Q_2 < 0$ and $|V_2/Q_2| < |Q_2/2|$.

With the help of equations (37*a*) and (38*a*), the corresponding ground-state energy and superpotential can be expressed as

$$\tilde{E}_0 = -\left[\frac{V_2}{Q_2} - \frac{Q_2}{2}\right]^2 - V_2, \tag{40}$$

$$W(x) = \left(\frac{V_2}{Q_2} - \frac{Q_2}{2}\right) + \frac{Q_2}{2}(1 - \tanh\alpha x). \tag{41}$$

By using the superpotential $W(x)$ given in equation (41), we can construct the following two supersymmetric partner potentials:

$$V_{\mathrm{eff}_+}(x) = W^2(x) + \frac{\mathrm{d}W(x)}{\mathrm{d}x} = \left(\frac{V_2}{Q_2} - \frac{Q_2}{2}\right)^2 + \frac{2V_2 - Q_2^2 - 2\alpha Q_2}{2}(1 - \tanh\alpha x) + \frac{Q_2^2 + 2\alpha Q_2}{4}(1 - \tanh\alpha x)^2, \tag{42a}$$

$$V_{\mathrm{eff}_-}(x) = W^2(x) - \frac{\mathrm{d}W(x)}{\mathrm{d}x} = \left(\frac{V_2}{Q_2} - \frac{Q_2}{2}\right)^2 + \frac{2V_2 - Q_2^2 + 2\alpha Q_2}{2}(1 - \tanh\alpha x) + \frac{Q_2^2 - 2\alpha Q_2}{4}(1 - \tanh\alpha x)^2. \tag{42b}$$

The partner potentials $V_{\mathrm{eff}_+}(x)$ and $V_{\mathrm{eff}_-}(x)$ satisfy the following relationship:

$$V_{\mathrm{eff}_+}(x, a_0) = V_{\mathrm{eff}_-}(x, a_1) + R(a_1), \tag{43}$$

where $a_0 = Q_2$, a_1 is a function of a_0, i.e., $a_1 = f(a_0) = a_0 + 2\alpha$, and $R(a_1)$ is independent of x, $R(a_1) = \left(\frac{V_2}{a_0} - \frac{a_0}{2}\right)^2 - \left(\frac{V_2}{a_1} - \frac{a_1}{2}\right)^2$. Equation (43) shows that the partner potentials $V_{\mathrm{eff}_+}(x)$ and $V_{\mathrm{eff}_-}(x)$ have are shape-invariant potentials in the senses of [23]. The energy spectra of the potential $V_{\mathrm{eff}_-}(x)$ are given by

$$\tilde{E}_0^{(-)} = 0, \tag{44a}$$

$$\begin{aligned}\tilde{E}_n^{(-)} &= \sum_{k=1}^{n} R(a_k) = R(a_1) + R(a_2) + \cdots + R(a_n) \\ &= \left(\frac{V_2}{a_0} - \frac{a_0}{2}\right)^2 - \left(\frac{V_2}{a_1} - \frac{a_1}{2}\right)^2 \\ &\quad + \left(\frac{V_2}{a_1} - \frac{a_1}{2}\right)^2 - \left(\frac{V_2}{a_2} - \frac{a_2}{2}\right)^2 + \cdots + \left(\frac{V_2}{a_{n-1}} - \frac{a_{n-1}}{2}\right)^2 - \left(\frac{V_2}{a_n} - \frac{a_n}{2}\right)^2 \\ &= \left(\frac{V_2}{a_0} - \frac{a_0}{2}\right)^2 - \left(\frac{V_2}{a_n} - \frac{a_n}{2}\right)^2 \\ &= \left(\frac{V_2}{Q_2} - \frac{Q_2}{2}\right)^2 - \left(\frac{V_2}{Q_2 + 2n\alpha} - \frac{Q_2 + 2n\alpha}{2}\right)^2,\end{aligned} \tag{44b}$$

where the quantum number $n = 0, 1, 2, \ldots$. From equations (35) and (42*b*), we get the following relation:

$$V_{\mathrm{eff}}(x) = -(V_1 \operatorname{sech}^2\alpha x + V_2 \tanh\alpha x) = V_{\mathrm{eff}_-}(x, a_0) + \tilde{E}_0. \tag{45}$$

By using equations (40), (44) and (45), we obtain the solution for $\tilde{E}$ in equation (33):

$$\tilde{E} = \tilde{E}_0 + \tilde{E}_n^{(-)} = -\left[\frac{V_2}{Q_2 + 2n\alpha}\right]^2 - \frac{(Q_2 + 2n\alpha)^2}{4}. \tag{46}$$

Substituting the expression of Q_2 given in equation (38*b*) into equation (46), we obtain the following expression:

$$\tilde{E} = -\frac{V_2^2}{4\alpha^2\left(n+\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{4V_1}{\alpha^2}}\right)^2} - \alpha^2\left(n+\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{4V_1}{\alpha^2}}\right)^2. \tag{47}$$

With the help of the expressions $V_1 = \eta^2 M_0^2$, $V_2 = -2\eta M_0^2$ and $\tilde{E} = E^2 - M_0^2(1+\eta^2)$ in equation (47), we obtain the relativistic energy spectrum for the imaginary potential (32) in the setting of the Dirac theory with the smooth step mass distribution,

$$E_n = \pm\left[M_0^2(1+\eta^2) - \frac{\eta^2 M_0^4}{\alpha^2(n+\delta_1)^2} - \alpha^2(n+\delta_1)^2\right]^{1/2}, \tag{48}$$

where $\delta_1 = \frac{1}{2}\left(1-\sqrt{1+\frac{4\eta^2M_0^2}{\alpha^2}}\right)$.

Substituting equation (47) into equation (33), we have the following equation:

$$\left[\frac{d^2}{dx^2} + \left(\eta^2 M_0^2 \operatorname{sech}^2\alpha x - 2\eta M_0^2 \tanh\alpha x\right)\right]\varphi(x) = \left[\frac{\eta^2 M_0^4}{\alpha^2\,(n+\delta_1)^2} + \alpha^2\,(n+\delta_1)^2\right]\varphi(x). \tag{49}$$

Introducing the new variable $z = -\tanh\alpha x$ and writing the wavefunction $\varphi(x)$ as $\varphi(x) = \left(\frac{1-z}{2}\right)^{-p}\left(\frac{1+z}{2}\right)^{-w}P(z)$, equation (49) can be reduced to the following equation satisfied by $P(z)$:

$$(1-z^2)\frac{d^2P}{dz^2} + [-2w+2p-(2-2p-2w)z]\frac{dP}{dz} + n(n-2p-2w+1)P = 0, \tag{50}$$

where $p = \frac{1}{2}\left[n+\delta_1-\frac{\eta M_0^2}{\alpha^2}\frac{1}{n+\delta_1}\right]$ and $w = \frac{1}{2}\left[n+\delta_1+\frac{\eta M_0^2}{\alpha^2}\frac{1}{n+\delta_1}\right]$. Equation (50) is the well-known differential equation satisfied by the Jacobi polynomials $P_n^{-2p,-2w}(z)$, hence the wavefunction $\varphi(x)$ can be expressed as

$$\varphi_n(x) = \left(\frac{1+\tanh\alpha x}{2}\right)^{-p}\left(\frac{1-\tanh\alpha x}{2}\right)^{-w} P_n^{-2p,-2w}(-\tanh\alpha x). \tag{51}$$

Applying the definition of the hyperbolic functions and making some algebraic manipulations, we may rewrite the wavefunction $\varphi(x)$ in the fashion

$$\varphi_n(x) = (\cosh\alpha x)^{(p+w)}\, e^{\alpha(w-p)x} P_n^{-2p,-2w}(-\tanh\alpha x). \tag{52}$$

With the help of equations (5) and (52), we obtain the unnormalized upper spinor wavefunction corresponding to energy level E_n,

$$\phi_n(x) = \sqrt{M_0(1+\eta\tanh\alpha x)}(\cosh\alpha x)^{(p+w)}\, e^{\alpha(w-p)x} P_n^{-2p,-2w}(-\tanh\alpha x). \tag{53}$$

In order to make the upper spinor component $\phi_n(x)$ satisfying the asymptotic boundary condition, $\phi_n(\pm\infty) = 0$, the exponent of $\cosh\alpha x$ must be smaller than zero, i.e., $p+w<0$ and $|p+w| > |w-p|$. Further, we can guarantee the real energy spectra E_n in equation (48) if and only if $M_0^2(1+\eta^2) \geqslant \frac{\eta^2M_0^4}{\alpha^2(n+\delta_1)^2} + \alpha^2(n+\delta_1)^2$. Consequently, we obtain the following restrictions for the quantum number n and the parameters α, η and M_0:

$$n < \sqrt{\frac{1}{4}+\frac{\eta^2M_0^2}{\alpha^2}} - \frac{1}{2}, \tag{54a}$$

$$\left|n+\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{\eta^2M_0^2}{\alpha^2}}\right| > \left|\frac{\eta M_0^2}{\alpha^2}\frac{1}{n+\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{\eta^2M_0^2}{\alpha^2}}}\right|, \tag{54b}$$

$$M_0(1+\eta) \geqslant \left| \frac{\eta M_0^2}{\alpha} \frac{1}{n+\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{\eta^2 M_0^2}{\alpha^2}}} + \alpha\left(n+\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{\eta^2 M_0^2}{\alpha^2}}\right)\right|. \tag{54c}$$

By using the differential and recursion properties of the Jacobi polynomials, we can determine the lower spinor wavefunction corresponding to energy level E_n from equation (3):

$$\theta_n(x) = \frac{1}{M_0(1+\eta\tanh\alpha x)}\left[\left(E_n + \mathrm{i}\left(\alpha(w-p) + \frac{n\alpha(p-w)}{n-p-w} + \alpha(-n+w+p)\tanh\alpha x\right)\right)\phi_n(x) - \frac{\mathrm{i}\alpha(n-2p)(n-2w)}{n-p-w}\phi_{n-1}(x)\right]. \tag{55}$$

4. Conclusion

In this work, we present a new method to construct the one-dimensional non-Hermitian imaginary potential with a real energy spectrum in the setting of the position-dependent effective mass Dirac equation with the vector-coupling scheme in 1+1 dimensions. In the first example of the mass distribution with linear and inversely linear forms in one spatial dimension, the problem is mapped into the exactly solvable Schrödinger-like equation problem with the isotonic oscillator. When the mass distribution is taken to have the form of the smooth step, the corresponding problem is mapped into the exactly solvable Schrödinger-like equation problem with the Rosen–Morse potential. The imaginary potential (11) possesses non-Hermitian *PT* symmetry. Although the imaginary potentials (11) and (32) are non-Hermitian, both of them have real bound-state energy spectra. Moreover, there are no bound states in the Schrödinger equation for the imaginary potential (32); this is due to the fact that the imaginary part of potential (32) behaves like a δ-like potential barrier. A similar case exists for a kink-like potential which is able to confine neutral fermions. Very recently, de Castro and Hott [27] found that there are exact bound-state solutions in the one-dimensional Dirac equation with the kink-like potential. However, this potential is characterized by the absence of bound states in the non-relativistic theory because it gives rise to a ubiquitous repulsive potential. With the help of the supersymmetric quantum mechanics approach and the function analysis method, we obtain the relativistic bound-state energy levels and the corresponding spinor wavefunctions for the imaginary potentials (11) and (32).

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